Rational solutions of the fourth and fifth Painlevé hierarchies

A. A. Grigor'ev (Minsk, Belarus)

We will consider four hierarchies of higher order analogues of the fourth (P4) and fifth (P5) Painleve equations. The necessary and sufficient conditions for having rational solutions will be presented. Also the algorithm for obtaining such solutions will be described.

The hierarchies obtained in [4] generalize the property of having an extended affine Weyl group of Bäcklund transformations. The hierarchy of (P4) is represented by the system of ordinary differential equations

$$\left\{\begin{array}{l} f_0' = f_0(f_1 - f_2 + f_3 - f_4 + \dots + f_{n-2} - f_{n-1}) + \alpha_0 \\ f_1' = f_1(f_2 - f_3 + f_4 - f_5 + \dots + f_{n-1} - f_0) + \alpha_1 \\ \dots \\ f_{n-1}' = f_{n-1}(f_0 - f_1 + f_2 - f_3 + \dots + f_{n-3} - f_{n-2}) + \alpha_{n-1}, \end{array}\right. \tag{1}$$

where α_i are such complex parameters, that $\sum_{i=0}^{n-1}\alpha_i = h \neq 0$, f_i are unknown functions and n is an odd number defining different members of the hierarchy. Further description of (1) and its properties connected to this paper could be found in [5], [8] and [9].

Another hierarchy of analogues of the (P4) was considered by Shabat [1], Veselov [2] and Adler [3]. It was obtained as the result of periodic closing of the dressing chain for linear Schrödinger equation.

$$\left\{\begin{array}{l} g_0' + g_1' = g_1^2 - g_0^2 + \alpha_0 \\ g_1' + g_2' = g_2^2 - g_1^2 + \alpha_1 \\ \dots \\ g_{n-1}' + g_0' = g_0^2 - g_{n-1}^2 + \alpha_{n-1}, \end{array}\right. \tag{2}$$

Here α_i are also such complex parameters that $\sum_{i=0}^{n-1}\alpha_i = h \neq 0$ and n is odd. With the scaling change of variables in (1) as also in (2) one can vary the value of h. Considering this we will further use $h = 1$ without the loss of generality.

As analogues of (P5) we will again consider the hierarchies of Noumi and Yamada from [4] with $(\sum_{i=0}^{2n+1}\alpha_i = h \neq 0)$ and arbitrary natural n defining the number of a member in the hierarchy. After change of variables it takes form

$$\left\{\begin{array}{l} z f_i' = z f_i \Phi_i - A_{Mod(i,2)} f_0 + \alpha_i C_{Mod(i,2)},\ i = \overline{0, 2n+1} \\ \sum\limits_{r=0}^{n} f_{2r} = C_0,\ \sum\limits_{s=0}^{n} f_{2s+1} = C_1 \end{array}\right. \tag{3}$$

where

$$\begin{array}{c} \Phi_i^{(n)} = \sum\limits_{1 \le r \le s \le n} f_{i+2r-1} f_{i+2s} - \sum\limits_{1 \le r \le s \le n} f_{i+2r} f_{i+2s+1}, \\ A_0^{(n)} = \sum\limits_{r=0}^{n} \alpha_{2r},\ A_1^{(n)} = \sum\limits_{s=0}^{n} \alpha_{2s+1}. \end{array} \tag{4}$$

Some properties of (3) connected to this paper are presented in [4], [10] and [11]. We will explain the correspondance of (3) to the systems of ordinary differential equations obtained by Shabat [1], Veselov [2] and Adler [3] with $(\sum_{i=0}^{2n+1}\alpha_i = h \neq 0)$.

$$\left\{\begin{array}{l} g_0' + g_1' = g_1^2 - g_0^2 + \alpha_0 \\ g_1' + g_2' = g_2^2 - g_1^2 + \alpha_1 \\ \dots \\ g_{2n+1}' + g_0' = g_0^2 - g_{2n+1}^2 + \alpha_{2n+1}, \end{array}\right. \tag{5}$$

Without the loss of generality we will again assume for (3) and (5) that $h = 1$.

When considering these hierarchies with real-valued parameters α_i, there exist a Bäcklund transformation, which leads to the system with $0 \le \alpha_i \le 1$ e.g. [8]. In this case the following sufficient conditions on α_i for existing of rational solutions are known: for the system (1) with odd n

$$\begin{array}{cccccccc} (\alpha_0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & .., & \alpha_n &) \\ \hline (1, & 0, & 0, & 0, & 0, & ..., & 0 &), \\ (\frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & 0, & 0, & ..., & 0 &), \\ (..., & ..., & ..., & ..., & ..., & ..., & ... &) \\ (\frac{1}{n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n}, & ..., & \frac{1}{n} &). \end{array} \quad \begin{array}{cccccccc} (f_0, & f_1, & f_2, & f_3, & f_4, & .., & f_n &) \\ \hline (t, & 0, & 0, & 0, & 0, & ..., & 0 &), \\ (\frac{t}{3}, & \frac{t}{3}, & \frac{t}{3}, & 0, & 0, & ..., & 0 &), \\ (..., & ..., & ..., & ..., & ..., & ..., & ... &) \\ (\frac{t}{n}, & \frac{t}{n}, & \frac{t}{n}, & \frac{t}{n}, & \frac{t}{n}, & ..., & \frac{t}{n} &). \end{array} \tag{6}$$

For the system (3) with $C_0 \neq 0$ and $C_1 \neq 0$ we also have

$$
\begin{array}{lcccccccl}
(\alpha_0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & .., & \alpha_{2n+1} &) & \\
\hline
(\alpha_0, & 1-\alpha_0, & 0, & 0, & 0, & ..., & 0 &), & \alpha_0 \in [0,1] \\
(\alpha_0, & \frac{1}{2}(1-2\alpha_0), & \alpha_0, & \frac{1}{2}(1-2\alpha_0), & 0, & ..., & 0 &), & \alpha_0 \in [0,\frac{1}{2}] \\
(..., & ..., & ..., & ..., & ..., & ..., & ... &) & \\
(\alpha_0, & \frac{1}{n+1}-\alpha_0, & \alpha_0, & \frac{1}{n+1}-\alpha_0, & \alpha_0, & ..., & \frac{1}{n+1}-\alpha_0 &), & \alpha_0 \in [0,\frac{1}{n+1}]
\end{array}
\tag{7}
$$

And if in the rows of the tables (6) and (7) we admit all possible permutations of all couples of zeroes this will describe the complete set of the parameters α_i, with which the hierarchies (1) and (3) admit rational solutions. Considering the relations between the both P4-hierarchies (1),(2) and also between the both P5-hierarchies (3), (5) the tables could be used for obtaining the conditions of having rational solutions for the systems (2) and (5).

Further we well consider two more hierarchies of the (P4). We well show that both of them admit the solution in the form of $1/x$. These hierarchies are the one that was introduced by Kudryashov in [6] as the compatibility condition

$$4PA_x + 2AP_x + \lambda Ay_x + 2\lambda yA_x - yy_xA + A_{xxx} + \lambda^2 + 2y_xA_x - Ay_{xx} - \lambda^2 A_x - y^2A_x - \lambda y = 0 \tag{8}$$

for the system of linear partial differential equations

$$\begin{cases} \Psi_{xx} = (\lambda - y(x))\Psi_x - P(x)\Psi \\ \lambda\Psi_\lambda = A(x,\lambda)\Psi_x + B(x,\lambda)\Psi, \end{cases} \tag{9}$$

where $A(x,\lambda)$ is a polynomial of λ. The degree of $A(x,\lambda)$ as a polynomial of λ corresponds to the member's number in the hierarchy. For example, the second member ($n = 2$) has the form:

$$
\begin{gathered}
(y'' - 2xy - 2y^3 - \beta)y^2y^{(4)} - \tfrac{1}{2}y^2\left(y^{(3)}\right)^2 + (2y^2 + 8y^3y' + 4xyy' - y'y'' + \beta y')yy^{(3)} - \\
-\tfrac{4}{3}y(y'')^3 + \left(3xy^2 + 3\beta y - \tfrac{3}{2}y^4 + \tfrac{3}{2}(y')^2\right)(y'')^2 + (\beta y^4 - 2y'y^2 - 12(y')^2y^3 - 2\beta^2 y + \\
+10xy^5 - 3\beta(y')^2 + 10y^7 - 4xy(y')^2 - 4\beta xy^2)y'' + 2(\beta - 4y^3)y^2y' + (4\beta xy + 8xy^4 + \\
+\tfrac{3}{2}\beta^2 + 12\beta y^3)(y')^2 - \tfrac{10}{3}y^{10} - 8xy^8 - 2\beta y^7 - 6x^2y^6 - 2\beta xy^5 + \\
+\left(\tfrac{1}{2}\beta^2 - 2 + 9\delta - \tfrac{4}{3}x^3\right)y^4 + \beta xy^2 + \tfrac{1}{3}\beta^3 y = 0,
\end{gathered}
\tag{10}
$$

In this equation β and δ are complex parameters.

And the last considered here analogues of the (P4) were obtained by Gordoa, Joshi and Pickering in [7] from nonisospectral scattering problem

$$\begin{cases} L_{n,x} - 2K_n - (u + 2\frac{g_{n+1}}{g_n})L_n = g_n - 2\alpha_n \\ L_nK_{n,x} + vL_n^2 + K_n^2 - L_{n,x}K_n(u + 2\frac{g_{n+1}}{g_n})L_nK_n = (\frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2 \end{cases}, \tag{11}$$

where

$$\begin{pmatrix} K_n \\ L_n \end{pmatrix} = B^{-1}\left[R^{n-1}U_x + g_n\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i=0}^{n-2}(\tfrac{-g_{n+1}}{g_n})^{n-i-1}R^iU_x\right] + 2(\tfrac{-g_{n+1}}{g_n})^{n-i-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad R = \tfrac{1}{2}\begin{pmatrix} \partial_x u\partial_x^{-1} & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix}. \tag{13}$$

In this problem α_n, β_n and g_i, $i = \overline{0, n+1}$ are complex parameters among which $g_{n-1} = 0$ and $g_n \neq 0$. The number n denotes the member's place in the resulting hierarchy.

The last result shows that all the considered P4-hierarchies have the common property: they assume the solution $1/x$ as also the original P4 does.

References

[1] Shabat A.B. *The infinite-dimensional dressing ynanical system*, Inverse Problems, 8(1992), 303-308.

[2] Veselov A.P. and Shabat A.B. *Dressing chains and the spectral theory of the Shroedinger operator*, Funct. Anal. Appl. 27(1993), 81-96.

[3] Adler V. E. *Nonlinear chains and Painlevé equations*, Physica. D 73 (1994), 335-351.

[4] M. Noumi, and Y. Yamada, *Higher order Painlevé equations of type* $A_l^{(1)}$ Funkcial. Ekvac. 41 483–503 1998.

[5] M. Noumi *Painlevé equations through symmery*, Translations of Mathematical Monographs Vol. 223, American Mathematical Society, 2004.

[6] N.A. Kudryashov *On the fourth Painleve hierarchy*, TMF 134-1, 2003

[7] P.R. Gordoa, N. Joshi and A. Pickering *On a Generalized 2+1 Dispersive Water Wave Hierarchy*, RIMS, Kyoto Univ., 37(2001), 327-347.

[8] (In Russian) Grigor'ev A.A. *On the solutions of the symmetric hierarchies of the fourth Painleve equation*, AMADE Proc., T-3, Institute of mathematics of NAS of Belarus, 2006, 41-48.

[9] Galina V. Filipuk, Peter A. Clarkson *The Symmetric Fourth Painleve Hierarchy and Associated Special Polynomials*, Studies in Applied Mathematics, Vol. 121, Issue 2, 157–188, 2008.

[10] (In Russian) Grigor'ev A.A. *On analytical properties of the analogue of the third and fifth Painleve equation.*, Vestnik PGU, Serie-C, 9 (2007), 37-47.

[11] (In Russian) Grigor'ev A.A. *On analytical properties of the hierarchies of the third and fifth Painleve equation.*, "Eruginskiye Chetiya-2007" Proc., Institute of mathematics of NAS of Belarus, 2007, 62-76.